

Weak Type (1, 1) Bounds for Some Operators Related to the Laplacian with Drift on Real Hyperbolic Spaces

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Abstract The setting of this work is the n -dimensional hyperbolic space $\mathbb{R}^+ \times \mathbb{R}^{n-1}$, where the Laplacian is given a drift in the \mathbb{R}^+ direction. We consider the operators defined by the horizontal Littlewood-Paley-Stein functions for the heat semigroup and the Poisson semigroup, and also the Riesz transforms of order 1 and 2. These operators are known to be bounded on L^p , $1 < p < \infty$, for the relevant measure. We show that most of the Littlewood-Paley-Stein operators and all the Riesz transforms are also of weak type (1, 1). But in some exceptional cases, we disprove the weak type (1, 1).

Keywords Littlewood-Paley-Stein function · Riesz transform · Laplacian with drift · Real hyperbolic space

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1 Introduction

Let (M, μ) be a σ -finite measure space, and let $\{T_t\}_{t>0}$ denote a symmetric diffusion semigroup on (M, μ) in the sense of [20]. The horizontal Littlewood-Paley-Stein function of order $k \in \mathbb{N}^+$ associated to $\{T_t\}_{t>0}$ is defined by

$$\left(\int_0^{+\infty} \left| s^k \frac{\partial^k}{\partial s^k} T_s f(x) \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}}, \quad x \in M,$$

for $f \in L^p(\mu)$, $1 \leq p < +\infty$, and the related maximal function of order $k \in \mathbb{N}$ is

$$\sup_{s>0} \left| s^k \frac{\partial^k}{\partial s^k} T_s f(x) \right|, \quad x \in M.$$

By the general Littlewood-Paley theory of Stein [20], these operators are bounded on $L^p(\mu)$ for all $1 < p < +\infty$. Some important examples of symmetric diffusion semigroups are the heat semigroup $e^{t\Delta}$ ($t > 0$) and the Poisson semigroup $e^{-t\sqrt{-\Delta}}$ ($t > 0$) associated to the Laplace-Beltrami operator Δ on a complete and stochastically complete (weighted) Riemannian manifold. For $f \in C_0^\infty$, set in the following

$$\begin{aligned} g_k(f)(x) &= \left(\int_0^{+\infty} \left| s^k \frac{\partial^k}{\partial s^k} e^{-s\sqrt{-\Delta}} f(x) \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}}, & G_k(f)(x) &= \sup_{s>0} \left| s^k \frac{\partial^k}{\partial s^k} e^{-s\sqrt{-\Delta}} f(x) \right|, \\ h_k(f)(x) &= \left(\int_0^{+\infty} \left| s^k \frac{\partial^k}{\partial s^k} e^{s\Delta} f(x) \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}}, & H_k(f)(x) &= \sup_{s>0} \left| s^k \frac{\partial^k}{\partial s^k} e^{s\Delta} f(x) \right|. \end{aligned}$$

Here $k \geq 1$ for g_k and h_k , but $k \geq 0$ for G_k and H_k .

In \mathbb{R}^n , it is obvious that g_k and h_k are not bounded on L^1 . But they are of weak type $(1, 1)$, as follows from the classical vector-valued singular integral operator theory; see for instance Stein [21, Ch. IV]. These results can be generalized to complete Riemannian manifolds satisfying the doubling volume property and an on-diagonal heat kernel upper estimate; see [3], where the weak type $(1, 1)$ of g_1 , g_2 and h_1 is proved. Moreover, it is not hard to see that the arguments of [3, pp. 50–52] are valid for g_k and h_k of higher order. In this setting, one can also show that the H_k are of weak type $(1, 1)$ by using basic properties of the centered Hardy-Littlewood maximal function.

If the manifolds considered have exponential volume growth, no doubling condition is satisfied. This situation is worse, since there is no adequate theory of singular integrals. There are some weak type $(1, 1)$ results for H_0 in that case; see for example [1, 2, 4, 14, 16] and references therein. To our knowledge, there exists only one result about g_k : in Anker's paper [1], the weak type $(1, 1)$ inequality of g_1 is obtained for noncompact symmetric spaces, and it is clear that the argument (see [1, pp. 290–291]) also works for g_k of higher order and in the setting of harmonic AN groups in the sense of [5]. The cases of h_k and H_k ($k \geq 1$) seem to be more difficult: there is no known result even for h_1 or H_1 .

The main purpose of this paper is to exhibit some manifolds of exponential volume growth in which h_1 and H_1 are of weak type $(1, 1)$. As a consequence, we give for symmetric spaces of noncompact type and rank one an affirmative answer to a problem left open in [1, Remark (1), pp. 278–279]. In the manifolds considered, we also treat Riesz transforms $\nabla(-\Delta)^{-1/2}$ and $(-\Delta)^{-1/2}\nabla$, and analogous second-order operators. For these, a few weak type $(1, 1)$ results have been obtained in the settings of noncompact symmetric spaces (see

for example [1]), harmonic AN groups (see [2]), affine groups (see for example [18] and [10]) and the Laplacian with drift on euclidean spaces (see [16]).

The setting of this paper will be a weighted manifold based on the real hyperbolic space \mathbb{H}^n of dimension $n \geq 2$. Here \mathbb{H}^n is considered as $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ endowed with the measure $d\mu(y, x) = y^{-n} dy dx$ and the distance

$$d((y, x), (y', x')) = \operatorname{arcosh} \frac{y^2 + (y')^2 + |x - x'|^2}{2yy}, \quad (y, x), (y', x') \in \mathbb{H}^n. \quad (1.1)$$

On \mathbb{H}^n we consider the vector fields

$$X_0 = y \frac{\partial}{\partial y}, \quad X_j = y \frac{\partial}{\partial x_j}, \quad 1 \leq j \leq n-1.$$

The gradient and its norm are given by

$$\nabla f = (X_0 f, \dots, X_{n-1} f) \quad \text{and} \quad |\nabla f|^2 = \sum_{j=0}^{n-1} |X_j f|^2.$$

The Laplacian on \mathbb{H}^n is

$$\Delta_{\mathbb{H}^n} = y^2 \frac{\partial^2}{\partial y^2} - (n-2)y \frac{\partial}{\partial y} + y^2 \Delta_{\mathbb{R}^{n-1}},$$

see [6, p. 176].

Given $\alpha \in \mathbb{R}$, we replace $d\mu$ by $d\mu_\alpha = y^\alpha d\mu$ and get $\mathbb{H}^{(n, \alpha)} = (\mathbb{H}^n, d\mu_\alpha)$, which is a weighted manifold as defined in [9, Definition 3.17, p. 67]. The corresponding Laplacian is obtained by adding to $\Delta_{\mathbb{H}^n}$ a drift term in the y coordinate,

$$\Delta_{\mathbb{H}^{(n, \alpha)}} = y^2 \frac{\partial^2}{\partial y^2} - (n-2-\alpha)y \frac{\partial}{\partial y} + y^2 \Delta_{\mathbb{R}^{n-1}},$$

cf. [9, pp. 252–253]. The bottom of the spectrum of $-\Delta_{\mathbb{H}^{(n, \alpha)}}$ in $L^2(\mathbb{H}^{(n, \alpha)})$ is $(n-1-\alpha)^2/4$, as follows from [9, Theorem 10.24, p. 292] combined with (2.4) and (2.7) below.

Notice that $\mathbb{H}^{(n, \alpha)}$ has exponential volume growth, which affects the behavior of the Hardy-Littlewood maximal function; see [11, 12] and [13]. Moreover, it is stochastically complete, see [9, Theorem 11.8, p. 303]. It follows that the operators g_k , G_k , h_k and H_k are bounded on $L^p(\mu_\alpha)$, $1 < p < \infty$.

For the Riesz transforms of any order on $\mathbb{H}^{(n, \alpha)}$, $\alpha \neq n-1$, the boundedness on $L^p(\mu_\alpha)$, $1 < p < \infty$, is proved by Lohoué and Mustapha [17], or can be deduced from this paper. We shall focus on the weak type $(1, 1)$ boundedness. The case $\alpha = n-1$ is special, since $\mathbb{H}^{(n, n-1)}$ is the affine group with the right-invariant Haar measure. In that case, explicit formulas for the operator kernels are available, and the weak type $(1, 1)$ boundedness properties of the Riesz transforms are known; see [19] and references there. Notice that this is the only case where $-\Delta_{\mathbb{H}^{(n, \alpha)}}$ has no spectral gap.

From now on, we exclude the affine group and assume that $\alpha \neq n-1$, except in Remark 3 below.

The following are our main results. The measure we use on $\mathbb{H}^{(n, \alpha)}$ is always $d\mu_\alpha$.

Theorem 1 *Let $n-1 \neq \alpha \in \mathbb{R}$. On $\mathbb{H}^{(n, \alpha)}$, the operators g_k , G_k ($k \geq 1$), h_1 , H_0 and H_1 are of weak type $(1, 1)$. For $k \geq 2$, h_k and H_k are not of weak type $(1, 1)$.*

Remark 1 It is easier to obtain the weak type $(1, 1)$ continuity for the first-order vertical Littlewood-Paley-Stein functions, i.e., those defined in terms of derivatives with respect to the x_i instead of t .

Remark 2 The results of Theorem 1 remain valid for harmonic AN groups in the sense of [5]. This is because for such a group the heat kernel has an explicit expression, given in [2, Theorem 5.9, p. 664], and an observation in [15, Section 6] makes it possible to control it in terms of the heat kernels of $\mathbb{H}^{(n,0)}$, with several values of n . One can then use [2, (5.26) Proposition, p. 667] and follow the method of our paper. The details are left to the reader. Notice that harmonic AN groups, as Riemannian manifolds, have constant negative Ricci curvature, see for example [2, p. 647]. They include all symmetric spaces of noncompact type and rank one, but most of them are not symmetric spaces.

Remark 3 It is worth observing that in the excluded case $\alpha = n - 1$, our methods, together with the Hopf-Dunford-Schwartz maximal ergodic theorem, can be used to show that H_k is of weak type $(1, 1)$. Moreover, our proof for the weak type $(1, 1)$ of G_k in Section 4 applies also in this case. But we do not know whether g_k and h_k are also of weak type $(1, 1)$ when $\alpha = n - 1$.

Theorem 2 Let $n - 1 \neq \alpha \in \mathbb{R}$.

(a) On $\mathbb{H}^{(n,\alpha)}$, the first-order Riesz transforms

$$X_j(-\Delta_{\mathbb{H}^{(n,\alpha)}})^{-\frac{1}{2}} \quad \text{and} \quad (-\Delta_{\mathbb{H}^{(n,\alpha)}})^{-\frac{1}{2}}X_j, \quad 0 \leq j \leq n - 1,$$

are of weak type $(1, 1)$.

(b) The same holds for the second-order Riesz transforms

$$X_i X_j (-\Delta_{\mathbb{H}^{(n,\alpha)}})^{-1}, \quad X_i (-\Delta_{\mathbb{H}^{(n,\alpha)}})^{-1} X_j, \quad (-\Delta_{\mathbb{H}^{(n,\alpha)}})^{-1} X_i X_j, \quad 0 \leq i, j \leq n - 1.$$

The structure of this paper is as follows. After some preliminaries in Section 2, we prove Theorem 2 in Section 3. The proof of Theorem 1 fills the remaining sections. First the case of G_k is treated in Section 4, in a more general setting. Then Section 5 contains the kernel estimates needed for the estimates of h_1 , H_0 , g_k and H_1 in Sections 7 and 8. Some relations between different Littlewood-Paley-Stein functions are obtained in Section 6. Finally, Section 9 proves the negative results in Theorem 1.

2 Notation and Preliminaries

Symbols like \sim and \lesssim will have their usual meaning, with implicit constants depending only on n and α .

It will be convenient to write

$$\rho(n, \alpha) = \frac{|n - 1 - \alpha|}{2} \quad \text{and} \quad \rho(n) = \frac{n - 1}{2}. \quad (2.2)$$

The subordination formula connects the Poisson and heat semigroups by

$$e^{-t\sqrt{-\Delta}} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-u}}{\sqrt{u}} e^{\frac{t^2}{4u}\Delta} du. \quad (2.3)$$

We denote by $p_t^{(n)}$ the heat kernel of \mathbb{H}^n and by $p_t^{(n,\alpha)}$ that of $\mathbb{H}^{(n,\alpha)}$. The two are related by

$$p_t^{(n,\alpha)}((y, x), (y', x')) = e^{\frac{\alpha}{2}(n-\frac{\alpha}{2}-1)t} (yy')^{-\frac{\alpha}{2}} p_t^{(n)}((y, x), (y', x')), \quad (2.4)$$

see [9, Theorem 9.15, p. 252].

To simplify notations in the sequel, we let

$$\mathcal{Y} = (y, x), \quad \mathcal{Y}' = (y', x') \quad \text{and} \quad r = d(\mathcal{Y}, \mathcal{Y}').$$

If a C^1 function f in $\mathbb{H}^{(n,\alpha)}$ is radial, or more generally if $f(\mathcal{Y}) = \tilde{f}(r)$ for some fixed $\mathcal{Y}' \in \mathbb{H}^n$, then

$$|\nabla_{\mathcal{Y}} f(\mathcal{Y})| = |\tilde{f}'(r)|, \quad (2.5)$$

as easily verified. Observe that

$$X_j = (X_j \cosh r) \frac{1}{\sinh r} \frac{\partial}{\partial r}, \quad 0 \leq j \leq n-1, \quad (2.6)$$

when these derivatives are applied to a function of r .

It is well known that the heat kernel of \mathbb{H}^n and the kernels of other operators which are given as functions of $\Delta_{\mathbb{H}^n}$ depend only on r . We shall write

$$p_t^{(n)}(r) = p_t^{(n)}(\mathcal{Y}, \mathcal{Y}').$$

Theorem 5.7.2, p. 179, of [6] says that for all $r \geq 0$ and $t > 0$

$$p_t^{(n)}(r) \sim t^{-\frac{n}{2}} (1+r+t)^{\frac{n-3}{2}} (1+r) e^{-\frac{(n-1)^2}{4}t - \frac{r^2}{4t} - \frac{n-1}{2}r}, \quad (2.7)$$

a result obtained by Davies and Mandouvalos [7].

Letting

$$p_t^{(1)}(r) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{r^2}{4t}}, \quad (2.8)$$

we have from [6, p. 178]

$$\frac{\partial}{\partial r} p_t^{(n)}(r) = -2\pi \sinh r e^{nt} p_t^{(n+2)}(r), \quad n = 1, 2, \dots \quad (2.9)$$

The space $\mathbb{H}^{(n,\alpha)}$ has the local doubling property. Indeed, the distance formula (1.1) implies that a ball $B(\mathcal{Y}, s)$ with $\mathcal{Y} = (y, x) \in \mathbb{H}^n$ and a small radius $s > 0$ is, up to a small error, given by the inequality

$$(y' - y)^2 + |x' - x|^2 < y^2 s^2.$$

It follows that

$$\mu_{\alpha}(B(\mathcal{Y}, s)) \sim y^{\alpha} s^n, \quad (2.10)$$

uniformly in \mathcal{Y} and $s \in (0, 1]$, and this implies the local doubling.

3 Weak Type (1, 1) Continuity of Riesz Transforms

We shall prove Theorem 2, and start with the operator $\nabla(-\Delta_{\mathbb{H}^{(n,\alpha)}})^{-\frac{1}{2}}$.

The kernel of the operator $(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}$ will be written simply as $(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}(\mathcal{Y}, \mathcal{Y}')$, and similarly for those of the Riesz transforms. We have because of (2.4)

$$\begin{aligned} (-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}(\mathcal{Y}, \mathcal{Y}') &= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-\frac{1}{2}} p_t^{(n,\alpha)}(\mathcal{Y}, \mathcal{Y}') dt \\ &= \frac{1}{\sqrt{\pi}} (yy')^{-\frac{\alpha}{2}} \int_0^{+\infty} t^{-\frac{1}{2}} e^{\frac{\alpha}{2}(n-\frac{\alpha}{2}-1)t} p_t^{(n)}(r) dt, \end{aligned} \quad (3.1)$$

Combining (3.1), (2.6) and (2.9), we get for $0 \leq j \leq n-1$

$$\begin{aligned} X_j(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}(\mathcal{Y}, \mathcal{Y}') &= \frac{1}{\sqrt{\pi}} X_j \left((yy')^{-\frac{\alpha}{2}} \right) \int_0^{+\infty} t^{-\frac{1}{2}} e^{\frac{\alpha}{2}(n-\frac{\alpha}{2}-1)t} p_t^{(n)}(r) dt \\ &\quad - 2\pi (yy')^{-\frac{\alpha}{2}} (X_j \cosh r) \frac{1}{\sqrt{\pi}} \\ &\quad \times \int_0^{+\infty} t^{-\frac{1}{2}} e^{[n+\frac{\alpha}{2}(n-\frac{\alpha}{2}-1)]t} p_t^{(n+2)}(r) dt. \end{aligned} \quad (3.2)$$

We apply (2.5) and (2.7), getting

$$\begin{aligned} \left| \nabla_{\mathcal{Y}}(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}(\mathcal{Y}, \mathcal{Y}') \right| &\lesssim (yy')^{-\frac{\alpha}{2}} \int_0^{+\infty} t^{-\frac{n+1}{2}} (1+r+t)^{\frac{n-3}{2}} (1+r) \\ &\quad \times e^{-\rho(n,\alpha)^2 t - \frac{r^2}{4t} - \frac{n-1}{2}r} dt \end{aligned} \quad (3.3)$$

$$\begin{aligned} &+ (yy')^{-\frac{\alpha}{2}} \sinh r \int_0^{+\infty} t^{-\frac{n+3}{2}} (1+r+t)^{\frac{n-1}{2}} (1+r) \\ &\quad \times e^{-\rho(n,\alpha)^2 t - \frac{r^2}{4t} - \frac{n+1}{2}r} dt. \end{aligned} \quad (3.4)$$

3.1 The Local Part

Considering the two integrals in (3.4) and (3.3), we observe that the exponents appearing in the integrands can be written

$$-\frac{\rho(n,\alpha)^2}{t} \left(t - \frac{r}{2\rho(n,\alpha)} \right)^2 - \rho(n,\alpha)r - \frac{n \pm 1}{2}r, \quad (3.5)$$

respectively, since $\rho(n,\alpha) > 0$. When $r \leq 1$, it follows that both integrands will be very small for $t \gg 1$ and for $t \ll r^2$. From this and (2.10), we conclude

$$\begin{aligned} \left| \nabla_{\mathcal{Y}}(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}(\mathcal{Y}, \mathcal{Y}') \right| &\lesssim (yy')^{-\frac{\alpha}{2}} \sinh r \int_{r^2 \lesssim t \lesssim 1} t^{-\frac{n+3}{2}} dt + (yy')^{-\frac{\alpha}{2}} \int_{r^2 \lesssim t \lesssim 1} t^{-\frac{n+1}{2}} dt \\ &\lesssim y^{-\alpha} r^{-n} \sim \frac{1}{\mu_{\alpha}(B(\mathcal{Y}, r))}, \quad 0 < r \leq 1. \end{aligned}$$

Now (3.2) implies that for $0 \leq i, j \leq n-1$

$$\begin{aligned} & X_i^{\mathcal{Y}'} X_j (-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}} (\mathcal{Y}, \mathcal{Y}') \\ &= \frac{1}{\sqrt{\pi}} X_i^{\mathcal{Y}'} X_j \left((yy')^{-\frac{\alpha}{2}} \right) \int_0^{+\infty} t^{-\frac{1}{2}} e^{\frac{\alpha}{2}(n-\frac{\alpha}{2}-1)t} p_t^{(n)}(r) dt \\ &+ \frac{1}{\sqrt{\pi}} X_j \left((yy')^{-\frac{\alpha}{2}} \right) \int_0^{+\infty} t^{-\frac{1}{2}} e^{\frac{\alpha}{2}(n-\frac{\alpha}{2}-1)t} X_i^{\mathcal{Y}'} p_t^{(n)}(r) dt \\ &- 2\pi \left\{ X_i^{\mathcal{Y}'} \left[(yy')^{-\frac{\alpha}{2}} (X_j \cosh r) \right] \right\} \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-\frac{1}{2}} e^{[n+\frac{\alpha}{2}(n-\frac{\alpha}{2}-1)]t} p_t^{(n+2)}(r) dt \\ &- 2\pi (yy')^{-\frac{\alpha}{2}} (X_j \cosh r) \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-\frac{1}{2}} e^{[n+\frac{\alpha}{2}(n-\frac{\alpha}{2}-1)]t} X_i^{\mathcal{Y}'} p_t^{(n+2)}(r) dt. \end{aligned}$$

Using (2.9) with n replaced by $n+2$ and (2.5), one obtains by similar computations

$$\left| \nabla_{\mathcal{Y}'} X_j (-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}} (\mathcal{Y}, \mathcal{Y}') \right| \lesssim \frac{1}{r \mu_\alpha(B(\mathcal{Y}, r))}, \quad 0 < r \leq 1, \quad 0 \leq j \leq n-1.$$

Thus the local parts of the Riesz kernels are standard Calderón-Zygmund kernels in $\mathbb{H}^{(n,\alpha)}$, which implies the weak type $(1, 1)$.

3.2 The Global Part

In this subsection, $r > 1$.

Invoking again the expression in (3.5), it easily follows that the order of magnitude of the integrals in (3.3) and (3.4) is determined by that part of the integral taken only over the interval

$$\left| t - \frac{r}{2\rho(n,\alpha)} \right| \lesssim r^{\frac{1}{2}}.$$

In that interval $1 + r + t \sim r$, and one finds

$$\left| \nabla_{\mathcal{Y}'} (-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}} (\mathcal{Y}, \mathcal{Y}') \right| \lesssim (yy')^{-\frac{\alpha}{2}} r^{-\frac{1}{2}} e^{-\rho(n,\alpha)r - \frac{n-1}{2}r}.$$

Here we suppress the power of r . What we need is then the following lemma.

Lemma 3 For $n-1 \neq \alpha \in \mathbb{R}$, the operator defined by integration against the kernel

$$T(\mathcal{Y}, \mathcal{Y}') = (yy')^{-\frac{\alpha}{2}} e^{-\rho(n,\alpha)r - \frac{n-1}{2}r}$$

is of weak type $(1, 1)$ with respect to μ_α .

Proof We can estimate T by applying (1.1) to the exponential factor, since $e^r \sim \cosh r$. The result will be

$$\begin{aligned} T(\mathcal{Y}, \mathcal{Y}') &\sim (yy')^{-\frac{\alpha}{2}} \left(\frac{yy'}{y^2 + y'^2 + |x - x'|^2} \right)^{\frac{|n-1-\alpha|}{2} + \frac{n-1}{2}} \\ &\sim (yy')^{\frac{|n-1-\alpha|}{2} + \frac{n-1-\alpha}{2}} (y + y' + |x - x'|)^{-|n-1-\alpha|-n+1}. \end{aligned} \quad (3.6)$$

Consider first the case $\alpha < n - 1$. Then if $f \in L^1(\mu_\alpha)$

$$\begin{aligned} |Tf(\mathcal{Y})| &\lesssim y^{n-1-\alpha} \int (y')^{n-1-\alpha} (y + y' + |x - x'|)^{\alpha-2(n-1)} |f(\mathcal{Y}')| d\mu_\alpha(\mathcal{Y}') \\ &\lesssim y^{n-1-\alpha} \int (y')^{n-1-\alpha} (y' + |x - x'|)^{\alpha-2(n-1)} |f(\mathcal{Y}')| d\mu_\alpha(\mathcal{Y}'). \end{aligned}$$

Since the last expression here is monotone in the variable y , we can argue as in Strömberg's paper [22]. This means observing for $\lambda > 0$ that $|Tf(\mathcal{Y})| > \lambda$ implies $y > y_0(x)$, where $y_0(x)$ satisfies

$$y_0(x)^{\alpha-n+1} \sim \frac{1}{\lambda} \int (y')^{n-1-\alpha} (y' + |x - x'|)^{\alpha-2(n-1)} |f(\mathcal{Y}')| d\mu_\alpha(\mathcal{Y}').$$

Thus $\mu_\alpha(\{\mathcal{Y}; |Tf(\mathcal{Y})| > \lambda\})$ is bounded by

$$\begin{aligned} \int dx \int_{y_0(x)}^\infty y^{\alpha-n} dy &\sim \int y_0(x)^{\alpha-n+1} dx \\ &= \frac{1}{\lambda} \int dx \int (y')^{n-1-\alpha} (y' + |x - x'|)^{\alpha-2(n-1)} |f(y', x')| d\mu_\alpha(y', x') \\ &= \frac{1}{\lambda} \int (y')^{n-1-\alpha} |f(y', x')| d\mu_\alpha(y', x') \int_{\mathbb{R}^{n-1}} (y' + |x - x'|)^{\alpha-2(n-1)} dx \\ &\lesssim \frac{\|f\|_1}{\lambda}, \end{aligned}$$

which ends this case.

In the case $\alpha > n - 1$, we get from (3.6)

$$T(\mathcal{Y}, \mathcal{Y}') \sim (y + y' + |x - x'|)^{-\alpha}$$

and thus

$$\begin{aligned} T(\mathcal{Y}, \mathcal{Y}') &\sim (y + y' + |x - x'|)^{-\alpha} \chi_{\{y' \geq y\}} + (y + y' + |x - x'|)^{-\alpha} \chi_{\{y' < y\}} \\ &= T_1(\mathcal{Y}, \mathcal{Y}') + T_2(\mathcal{Y}, \mathcal{Y}'), \end{aligned}$$

say.

The T_1 part here gives rise to a strong type $(1, 1)$ operator, since

$$\int T_1(\mathcal{Y}, \mathcal{Y}') d\mu_\alpha(\mathcal{Y}) \leq \int_0^{y'} y^{\alpha-n} dy \int_{\mathbb{R}^{n-1}} (y' + |x - x'|)^{-\alpha} dx \sim 1.$$

The T_2 part requires a longer argument. We first observe that $T_2 \sim y^{-\alpha}$ for $|x - x'| < y$. If $2^{j-1}y \leq |x - x'| < 2^j y$ for some $j \in \{1, 2, \dots\}$, one similarly has $T_2 \sim 2^{-\alpha j} y^{-\alpha}$. As a result, we see that in all of \mathbb{H}^n

$$T_2 \lesssim \sum_0^\infty Q_j,$$

where

$$Q_j(\mathcal{Y}, \mathcal{Y}') = 2^{-\alpha j} y^{-\alpha} \chi_{\{y' < y, |x - x'| < 2^j y\}}.$$

The following lemma will end the proof of Lemma 3, since it will allow summation in j . \square

Lemma 4 Assume that $\alpha > n - 1$. For $j = 0, 1, 2, \dots$, the operator defined by integration against the kernel Q_j is of weak type $(1, 1)$ with a constant which is $O(2^{-\varepsilon j})$ as $j \rightarrow \infty$, for some $\varepsilon > 0$.

Proof We fix $j \in \{0, 1, 2, \dots\}$, let $0 \leq f \in L^1(\mu_\alpha)$ and take $\lambda > 0$. If (y, x) is a point in the level set defined by

$$\int Q_j((y, x), (y', x')) f(y', x') d\mu_\alpha(y', x') > \lambda, \quad (3.7)$$

then

$$2^{-\alpha j} y^{-\alpha} \int_{Z_{y,x}} f d\mu_\alpha > \lambda, \quad (3.8)$$

where $Z_{y,x}$ is the cylinder

$$Z_{y,x} = \{(y', x') : y' < y, |x - x'| < 2^j y\}.$$

Consider the family \mathcal{Z} of all cylinders $Z_{y,x}$ with $(y, x) \in \mathbb{H}^n$ which verify (3.8). We will mimic the ordinary proof of the weak type $(1, 1)$ inequality for the standard maximal function in \mathbb{R}^n . Notice that $\mu_\alpha(Z_{y,x}) \sim 2^{(n-1)j} y^\alpha$, and that for any $Z_{y,x} \in \mathcal{Z}$ the inequality (3.8) implies

$$2^{(n-1)j} y^\alpha < 2^{-(\alpha-n+1)j} \frac{1}{\lambda} \int_{Z_{y,x}} f d\mu_\alpha \leq 2^{-(\alpha-n+1)j} \frac{1}{\lambda} \int f d\mu_\alpha < \infty. \quad (3.9)$$

We shall define recursively a sequence $(Z^k)_1^\infty$ of pairwise disjoint cylinders in \mathcal{Z} . At each step, we shall choose a $Z_{y,x}$ with y essentially as large as possible, among the cylinders disjoint with those already chosen. Let first $Z^1 = Z_{y_1, x_1}$ be any cylinder in \mathcal{Z} verifying

$$y_1 > \frac{1}{2} \sup \{y : \exists x \in \mathbb{R}^{n-1} \text{ such that } Z_{y,x} \in \mathcal{Z}\}.$$

From Eq. 3.9 we see that this supremum is finite. Assuming Z^1, \dots, Z^{k-1} already defined, we let $Z^k = Z_{y_k, x_k}$ be any cylinder in \mathcal{Z} disjoint with Z^1, \dots, Z^{k-1} and verifying

$$y_k > \frac{1}{2} \sup \{y : \exists x \in \mathbb{R}^{n-1} \text{ such that } Z_{y,x} \in \mathcal{Z} \text{ and } Z_{y,x} \text{ is disjoint with } Z^1, \dots, Z^{k-1}\}. \quad (3.10)$$

Should the set here be empty, the procedure stops. Since the Z^k are pairwise disjoint, Eq. 3.9 implies

$$\sum_k \mu_\alpha(Z^k) < 2^{-(\alpha-n+1)j} \frac{1}{\lambda} \sum_k \int_{Z^k} f d\mu_\alpha \leq 2^{-(\alpha-n+1)j} \frac{1}{\lambda} \int f d\mu_\alpha < \infty. \quad (3.11)$$

In particular, both $\mu_\alpha(Z^k)$ and y_k will tend to 0 as $k \rightarrow \infty$, and so the supremum in Eq. 3.10 also tends to 0. Now assume $Z_{y,x}$ is a cylinder in \mathcal{Z} which is not among the Z^k . Then $Z_{y,x}$ must intersect some Z^k , since otherwise y would be less than the supremum in Eq. 3.10 for each k . Let Z^k be the first cylinder in the sequence which intersects $Z_{y,x}$. It follows from the choice of Z^k that $y_k > y/2$.

But then the enlarged cylinder $3Z^k = Z_{3y_k, x_k}$ will contain the point (y, x) . That means that the union set $\cup_k 3Z^k$ contains all points (y, x) in the level set defined by Eq. 3.7. The μ_α measure of this level set is then at most

$$\mu_\alpha(\cup_k 3Z^k) \leq \sum_k \mu_\alpha(3Z^k) \sim \sum_k \mu_\alpha(Z^k) \leq 2^{-(\alpha-n+1)j} \lambda^{-1} \int f d\mu_\alpha, \quad (3.12)$$

where the last step comes from Eq. 3.11.

Lemma 4 is proved with $\varepsilon = \alpha - n + 1$, and Lemma 3 follows. With this, Theorem 2(a) is proved for the the Riesz transforms $X_j(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}$.

To deal with the operators $(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}X_j$, we first verify their L^p boundedness. Taking adjoints in $L^2(d\mu_\alpha)$, we have

$$X_0^* = -X_0 - (\alpha - n + 1) \quad \text{and} \quad X_j^* = -X_j, \quad 1 \leq j \leq n-1. \quad (3.13)$$

Since the spectral gap of $-\Delta_{\mathbb{H}(n,\alpha)}$ is $\rho^2(n, \alpha) \neq 0$, it follows that $(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}$ is bounded on L^p , $1 < p < +\infty$. Using the L^p boundedness of $X_j(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}$ and its adjoint, we conclude that $(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}X_j$ is also bounded on L^p , $1 < p < +\infty$. We must prove the weak type $(1, 1)$.

For $1 \leq j \leq n-1$, one finds that

$$(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}X_j(\mathcal{V}, \mathcal{V}') = -y' \frac{\partial}{\partial x'_j} (-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}(\mathcal{V}, \mathcal{V}'). \quad (3.14)$$

We will thus get an expression for this kernel analogous to Eq. 3.2, with only the last summand, and with $X_j \cosh r = y \partial \cosh r / \partial x_j$ replaced by $-y' \partial \cosh r / \partial x'_j$.

The case $j = 0$ is only slightly more complicated, and one finds

$$\begin{aligned} (-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}X_0(\mathcal{V}, \mathcal{V}') &= -y' \frac{\partial}{\partial y'} (-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}(\mathcal{V}, \mathcal{V}') \\ &\quad - (\alpha + 1 - n)(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}(\mathcal{V}, \mathcal{V}'). \end{aligned} \quad (3.15)$$

Here we get two terms like those in Eq. 3.2, but with $X_0 \cosh r = y \partial \cosh r / \partial y$ replaced by $-y' \partial \cosh r / \partial y'$ and with $X_0(yy')^{-\frac{\alpha}{2}} = -\frac{\alpha}{2}(yy')^{-\frac{\alpha}{2}}$ replaced by $(n-1-\alpha/2)(yy')^{-\frac{\alpha}{2}}$.

It is now easy to verify that the arguments given above for the local and global parts remain valid for the operators $(-\Delta_{\mathbb{H}(n,\alpha)})^{-\frac{1}{2}}X_j$ and lead to the weak type $(1, 1)$ estimate.

Part (a) of Theorem 2 is proved.

For Part (b), one can follow the pattern of the proof of Part (a), and we leave the details to the reader.

This ends the proof of Theorem 2. \square

4 Weak Type $(1, 1)$ of G_k for a General Symmetric Diffusion Semigroup

Let $\{e^{t\Delta}\}_{t>0}$ be a symmetric diffusion semigroup and $\{e^{-t\sqrt{-\Delta}}\}_{t>0}$ the corresponding Poisson semigroup. Stein's argument (see [20, pp. 48–49]) leads to the weak type $(1, 1)$ inequality for G_k . Indeed, the subordination formula (2.3) can be written as

$$e^{-t\sqrt{-\Delta}} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{1}{u} \left(\frac{t}{2\sqrt{u}} e^{-\frac{t^2}{4u}} \right) e^{u\Delta} du = \int_0^{+\infty} \frac{1}{u} \phi \left(\frac{t}{2\sqrt{u}} \right) e^{u\Delta} du, \quad (4.1)$$

where

$$\phi(s) = \frac{s}{\sqrt{\pi}} e^{-s^2}. \quad (4.2)$$

Using integration by parts, we obtain

$$e^{-t\sqrt{-\Delta}} = \int_0^{+\infty} \left(\frac{1}{u} \int_0^u e^{s\Delta} ds \right) \frac{1}{u} \left[\phi \left(\frac{t}{2\sqrt{u}} \right) + \phi' \left(\frac{t}{2\sqrt{u}} \right) \frac{t}{4\sqrt{u}} \right] du.$$

Letting

$$\psi(s) = \phi(s) + \frac{1}{2}\phi'(s)s = \left(\frac{3}{2} - s^2 \right) \frac{s}{\sqrt{\pi}} e^{-s^2},$$

we have

$$e^{-t\sqrt{-\Delta}} = \int_0^{+\infty} \left(\frac{1}{u} \int_0^u e^{s\Delta} ds \right) \frac{1}{u} \psi \left(\frac{t}{2\sqrt{u}} \right) du.$$

Then for $k \in \mathbb{N}$, $t > 0$ and $f \in L^1$

$$\begin{aligned} \left| t^k \frac{\partial^k}{\partial t^k} e^{-t\sqrt{-\Delta}} f \right| &= \left| \int_0^{+\infty} \left(\frac{1}{u} \int_0^u e^{s\Delta} f ds \right) \frac{1}{u} \psi^{(k)} \left(\frac{t}{2\sqrt{u}} \right) \left(\frac{t}{2\sqrt{u}} \right)^k du \right| \\ &\leq \sup_{v>0} \left| \frac{1}{v} \int_0^v e^{s\Delta} f ds \right| \int_0^{+\infty} \left| \frac{1}{u} \psi^{(k)} \left(\frac{t}{2\sqrt{u}} \right) \left(\frac{t}{2\sqrt{u}} \right)^k \right| du. \end{aligned}$$

The change of variable $\lambda = \frac{t}{2\sqrt{u}}$ shows that the last integral equals

$$A_k = 2 \int_0^{+\infty} \left| \lambda^2 \psi^{(k)}(\lambda) \lambda^k \right| \lambda^{-3} d\lambda < +\infty.$$

Thus

$$G_k(f) \leq A_k \sup_{v>0} \left| \frac{1}{v} \int_0^v e^{s\Delta} f ds \right|,$$

and the Hopf-Dunford-Schwartz ergodic theorem implies the $L^1 \rightarrow L^{1,\infty}$ boundedness of G_k . \square

5 Sharp Estimates for $\frac{\partial^k}{\partial t^k} p_t^{(n,\alpha)}$

Set

$$K(n, \alpha; t, r) = (yy')^{\frac{\alpha}{2}} p_t^{(n,\alpha)}(r) = e^{\frac{\alpha}{2}(n-\frac{\alpha}{2}-1)t} p_t^{(n)}(r), \quad (5.1)$$

where the second equality comes from Eq. 2.4.

It is well known (see e.g. [6, formula (5.7.4), p. 178]) that

$$p_t^{(n)}(r) = \sqrt{2} e^{\frac{2n-1}{4}t} \int_r^{+\infty} p_t^{(n+1)}(s) \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} ds, \quad n = 1, 2, \dots, \quad (5.2)$$

for all $t > 0$ and $r \geq 0$. From Eqs. 5.1 and 2.7, we conclude

$$K(n, \alpha; t, r) \sim t^{-\frac{1}{2}} \frac{1+r}{t} \left(1 + \frac{1+r}{t} \right)^{\frac{n-3}{2}} \exp \left(-\rho(n)r - \frac{r^2}{4t} - \rho^2(n, \alpha)t \right) \quad (5.3)$$

$$\begin{aligned} &= t^{-\frac{1}{2}} \frac{1+r}{t} \left(1 + \frac{1+r}{t} \right)^{\frac{n-3}{2}} \\ &\quad \times \exp \left(-(\rho(n) + \rho(n, \alpha))r - t \left(\frac{r}{2t} - \rho(n, \alpha) \right)^2 \right). \end{aligned} \quad (5.4)$$

After some computations, we get from Eqs. 5.1, 2.9 and 2.8

$$K(2n+1, \alpha; t, r) = \left(-\frac{1}{2\pi} \frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^n \left[\frac{1}{\sqrt{4\pi t}} e^{-\frac{r^2}{4t} - \rho^2(2n+1, \alpha)t} \right].$$

This yields

$$\frac{\partial}{\partial t} K(2n+1, \alpha; t, r) = \left\{ \left(\frac{r^2}{4t^2} - \rho^2(2n+1, \alpha) \right) - \frac{1}{2t} \right\} K(2n+1, \alpha; t, r), \quad (5.5)$$

and

$$\frac{\partial^2}{\partial t^2} K(2n+1, \alpha; t, r) = \left\{ -\frac{1}{2} \frac{r^2}{t^2} \cdot \frac{1}{t} + \frac{1}{2} \left(\frac{1}{t} \right)^2 + \left[\left(\frac{r^2}{4t^2} - \rho^2(2n+1, \alpha) \right) - \frac{1}{2t} \right]^2 \right\} \times K(2n+1, \alpha; t, r). \quad (5.6)$$

In general, let $P_j(h_1, h_2, h_3)$ be the real homogeneous polynomials of degree j on \mathbb{R}^3 defined recursively for $j = 0, 1, \dots$ by

$$P_0 = 1, \quad P_{j+1} = \left(h_1 - \frac{1}{2} h_2 \right) P_j - \frac{1}{2} h_2 h_3 \frac{\partial P_j}{\partial h_1} - h_2^2 \frac{\partial P_j}{\partial h_2} - 2h_2 h_3 \frac{\partial P_j}{\partial h_3}.$$

With $j = 0, 1, \dots$, we then have

$$\frac{\partial^j}{\partial t^j} K(2n+1, \alpha; t, r) = P_j \left(\frac{r^2}{4t^2} - \rho^2(2n+1, \alpha), \frac{1}{t}, \frac{r^2}{t^2} \right) K(2n+1, \alpha; t, r). \quad (5.7)$$

For even dimensions, Eqs. 5.5 and 5.7 hold only with an error term, which we shall estimate. From Eq. 5.2, we get for $j = 0, 1, \dots$

$$\begin{aligned} \frac{\partial^j}{\partial t^j} K(2n, \alpha; t, r) &= \sqrt{2} \int_r^{+\infty} P_j \left(\frac{s^2}{4t^2} - \rho^2(2n, \alpha), \frac{1}{t}, \frac{s^2}{t^2} \right) \\ &\quad \times \frac{e^{\frac{4n-1-2\alpha}{4}t} K(2n+1, \alpha; t, s) \sinh s}{\sqrt{\cosh s - \cosh r}} ds. \end{aligned} \quad (5.8)$$

Proposition 5 For $r \geq 0$ and any $t > 0$, we have

$$\left| \frac{\partial}{\partial t} K(2n, \alpha; t, r) - \left[\left(\frac{r^2}{4t^2} - \rho^2(2n, \alpha) \right) - \frac{1}{2t} \right] K(2n, \alpha; t, r) \right| \lesssim \frac{1}{t} K(2n, \alpha; t, r) \quad (5.9)$$

and

$$\begin{aligned} &\left| \frac{\partial^2}{\partial t^2} K(2n, \alpha; t, r) - P_2 \left(\frac{r^2}{4t^2} - \rho^2(2n, \alpha), \frac{1}{t}, \frac{r^2}{t^2} \right) K(2n, \alpha; t, r) \right| \\ &\lesssim \left\{ \frac{1}{t} \left| \frac{r^2}{4t^2} - \rho^2(2n, \alpha) \right| + \frac{1}{t^2} \right\} K(2n, \alpha; t, r), \end{aligned} \quad (5.10)$$

and with $j = 1, 2, \dots$ also

$$\left| \frac{\partial^j}{\partial t^j} K(2n, \alpha; t, r) \right| \lesssim \left[\left| \frac{r^2}{4t^2} - \rho^2(2n, \alpha) \right|^j + \frac{1}{t^j} + \left(\frac{r}{t} \right)^{2j} \right] K(2n, \alpha; t, r), \quad (5.11)$$

Proof We first give the proof of Eq. 5.9. According to Eq. 5.8, we can write

$$\begin{aligned} & \frac{\partial}{\partial t} K(2n, \alpha; t, r) - \left[\left(\frac{r^2}{4t^2} - \rho^2(2n, \alpha) \right) - \frac{1}{2t} \right] K(2n, \alpha; t, r) \\ &= \sqrt{2} \int_r^{+\infty} \left[P_1 \left(\frac{s^2}{4t^2} - \rho^2(2n, \alpha), \frac{1}{t}, \frac{s^2}{t^2} \right) - P_1 \left(\frac{r^2}{4t^2} - \rho^2(2n, \alpha), \frac{1}{t}, \frac{r^2}{t^2} \right) \right] \\ & \quad \times e^{\frac{4n-1-2\alpha}{4}t} \frac{K(2n+1, \alpha; t, s) \sinh s}{\sqrt{\cosh s - \cosh r}} ds \\ &= \frac{\sqrt{2}}{4} \int_r^{+\infty} \frac{s^2 - r^2}{t^2} e^{\frac{4n-1-2\alpha}{4}t} \frac{K(2n+1, \alpha; t, s) \sinh s}{\sqrt{\cosh s - \cosh r}} ds, \end{aligned} \quad (5.12)$$

and Eq. 5.9 will be a consequence of the case $j = 1$ of the following lemma. \square

Lemma 6 For all $r > 0$, $t > 0$ and $j = 1, 2, \dots$

$$\int_r^{+\infty} \left(\frac{s^2 - r^2}{t^2} \right)^j e^{\frac{4n-1-2\alpha}{4}t} \frac{K(2n+1, \alpha; t, s) \sinh s}{\sqrt{\cosh s - \cosh r}} ds \lesssim t^{-j} K(2n, \alpha; t, r), \quad (5.13)$$

where the implicit constant depends only on j , α and n .

Before the proof of this lemma, we finish that of Proposition 5. The arguments for Eq. 5.10 are similar to those for Eq. 5.9 just given. Instead of the factor $(s^2 - r^2)/t^2$ in Eq. 5.12, we will now have

$$\begin{aligned} & \left| P_2 \left(\frac{s^2}{4t^2} - \rho^2(2n, \alpha), \frac{1}{t}, \frac{s^2}{t^2} \right) - P_2 \left(\frac{r^2}{4t^2} - \rho^2(2n, \alpha), \frac{1}{t}, \frac{r^2}{t^2} \right) \right| \\ &= \left| -\frac{1}{2} \frac{1}{t} \frac{s^2 - r^2}{t^2} + \frac{s^2 - r^2}{4t^2} \cdot 2 \left\{ \left(\frac{r^2}{4t^2} - \rho^2(2n, \alpha) \right) - \frac{1}{2t} \right\} + \left(\frac{s^2 - r^2}{4t^2} \right)^2 \right| \\ &\lesssim \left(\left| \frac{r^2}{4t^2} - \rho^2(2n, \alpha) \right| + \frac{1}{t} \right) \frac{s^2 - r^2}{t^2} + \left(\frac{s^2 - r^2}{t^2} \right)^2, \end{aligned}$$

and Lemma 6 can be applied again.

To prove Eq. 5.11, one uses the estimate

$$\begin{aligned} \left| P_j \left(\frac{s^2}{4t^2} - \rho^2(2n, \alpha), \frac{1}{t}, \frac{s^2}{t^2} \right) \right| &= \left| P_j \left(\frac{s^2 - r^2}{4t^2} + \frac{r^2}{4t^2} - \rho^2(2n, \alpha), \frac{1}{t}, \frac{s^2 - r^2}{t^2} + \frac{r^2}{t^2} \right) \right| \\ &\lesssim \left| \frac{r^2}{4t^2} - \rho^2(2n, \alpha) \right|^j + \frac{1}{t^j} + \left(\frac{r}{t} \right)^{2j} + \left(\frac{s^2 - r^2}{t^2} \right)^j, \end{aligned}$$

valid for $s \geq r \geq 0$, since P_j is a homogeneous polynomial of degree j . We omit the details. This ends the proof of Proposition 5. \square

Proof of Lemma 6 We start with the case $r \geq 1$. Since for $s \geq r \geq 1$,

$$\cosh s - \cosh r \geq \cosh s - \cosh \frac{r+s}{2} \geq \frac{s-r}{2} \sinh \frac{r+s}{2} \gtrsim (s-r)e^{\frac{r+s}{2}},$$

we have

$$\frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \lesssim \frac{1}{\sqrt{s-r}} e^{\frac{3s-r}{4}}.$$

In the left-hand side of Eq. 5.13 we now apply this estimate, and replace the values of the function K by the expressions obtained from Eq. 5.3. The result is that Eq. 5.13 will be a consequence of the inequality

$$\int_r^{+\infty} \left(\frac{s^2 - r^2}{t^2} \right)^j s \left(1 + \frac{s}{t} \right)^{n-1} \frac{1}{\sqrt{s-r}} e^{Q(s,r;t)} ds \lesssim \frac{1}{t^j} r \left(1 + \frac{r}{t} \right)^{n-\frac{3}{2}}, \quad (5.14)$$

where

$$\begin{aligned} Q(s, r; t) = & \frac{4n-1-2\alpha}{4} t - \rho(2n+1)s - \frac{s^2}{4t} - \rho^2(2n+1, \alpha)t \\ & + \frac{3s-r}{4} + \rho(2n)r + \frac{r^2}{4t} + \rho^2(2n, \alpha)t. \end{aligned}$$

Since $\rho(2n) = \rho(2n+1) - \frac{1}{2}$ and, as easily verified,

$$\rho^2(2n+1, \alpha) - \frac{4n-1-2\alpha}{4} = \rho^2(2n, \alpha),$$

one finds that

$$Q(s, r; t) = - \left(n - \frac{3}{4} + \frac{s+r}{4t} \right) (s-r) \leq -\frac{1}{4} \left(1 + \frac{r}{t} \right) (s-r)$$

for $s > r \geq 1$ and all $t > 0$. Hence, Eq. 5.14 will follow from the estimate

$$\int_r^{+\infty} s^{j+1} \left(1 + \frac{s}{t} \right)^{n-1} (s-r)^{j-\frac{1}{2}} e^{-\frac{1}{4}(1+\frac{r}{t})(s-r)} ds \lesssim t^j r \left(1 + \frac{r}{t} \right)^{n-\frac{3}{2}}.$$

Using the simple inequality $s e^{-\varepsilon(s-r)} \lesssim r$ for $s > r \geq 1$ and $\varepsilon > 0$, we can reduce this to

$$\int_r^{+\infty} r^{j+1} \left(1 + \frac{r}{t} \right)^{n-1} (s-r)^{j-\frac{1}{2}} e^{-\frac{1}{5}(1+\frac{r}{t})(s-r)} ds \lesssim t^j r \left(1 + \frac{r}{t} \right)^{n-\frac{3}{2}},$$

which is easily justified by means of the change of variables $u = (1 + \frac{r}{t})(s-r)$.

Consider now the case $0 \leq r < 1$. Then

$$\frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \sim \frac{s}{\sqrt{s^2 - r^2}} e^{\frac{s}{2}}, \quad s > r,$$

as one sees by considering small and large values of s . Instead of Eq. 5.14, we now need to prove that

$$\int_r^{+\infty} \left(\frac{s^2 - r^2}{t^2} \right)^j (1+s) \left(1 + \frac{1+s}{t} \right)^{n-1} \frac{s}{\sqrt{s^2 - r^2}} e^{\tilde{Q}(s,r;t)} ds \lesssim \frac{1}{t^j} \left(1 + \frac{1}{t} \right)^{n-\frac{3}{2}}, \quad (5.15)$$

for $0 < r \leq 1$, $t > 0$, where

$$\tilde{Q}(s, r; t) = Q(s, r; t) - \frac{3s-r}{4} + \frac{s}{2} = - \left(n - \frac{1}{2} \right) (s-r) - \frac{s^2 - r^2}{4t} \leq -s - \frac{s^2 - r^2}{4t} + O(1).$$

Thus Eq. 5.15 will follow from

$$\int_r^{+\infty} (s^2 - r^2)^{j-\frac{1}{2}} s (1+s) \left(1 + \frac{1+s}{t} \right)^{n-1} e^{-s - \frac{s^2 - r^2}{4t}} ds \lesssim t^j \left(1 + \frac{1}{t} \right)^{n-\frac{3}{2}}.$$

The case $t \geq 1$ of this inequality is trivial, since the integral is uniformly bounded for these t . For $0 < t < 1$ we have $(1+s) \left(1 + \frac{1+s}{t}\right)^{n-1} e^{-s} \lesssim t^{1-n}$. After the change of variables $u = (s^2 - r^2)/4t$, the integral can then be estimated by constant times

$$\int_0^{+\infty} (4tu)^{j-\frac{1}{2}} t^{1-n} e^{-u} 2t \, du \lesssim t^{j-n+\frac{3}{2}} \sim t^j \left(1 + \frac{1}{t}\right)^{n-\frac{3}{2}},$$

as desired. The lemma is proved. \square

To summarize, it follows from Eqs. 5.9 and 5.5 that for all $n \geq 2$

$$\left| \frac{\partial}{\partial t} K(n, \alpha; t, r) \right| \lesssim \left[\left| \frac{r^2}{4t^2} - \rho^2(n, \alpha) \right| + \frac{1}{t} \right] K(n, \alpha; t, r), \quad r \geq 0, \quad t > 0. \quad (5.16)$$

Further, Eqs. 5.10 and 5.6 easily imply that

$$\frac{\partial^2}{\partial t^2} K(n, \alpha; t, r) \sim -\frac{1}{r} K(n, \alpha; t, r), \quad \forall r \gg 1 \text{ with } \sqrt{r} \left| \frac{r^2}{4t^2} - \rho^2(n, \alpha) \right| \ll 1. \quad (5.17)$$

From Eqs. 5.11 and 5.7, we obtain

$$\left| \frac{\partial^j}{\partial t^j} K(n, \alpha; t, r) \right| \lesssim \left[1 + \frac{1}{t^j} + \left(\frac{r}{t} \right)^{2j} \right] K(n, \alpha; t, r), \quad r \geq 0, \quad t > 0. \quad (5.18)$$

6 Comparison Between g_k , G_k , H_k and h_k on $\mathbb{H}^{(n, \alpha)}$

Proposition 7 For all $k \in \mathbb{N}^+$ and $f \in L^1$, we have

$$h_k(f)(\mathcal{Y}) \leq \sqrt{\frac{1}{2k-1}} h_{k+1}(f)(\mathcal{Y}), \quad g_k(f)(\mathcal{Y}) \leq \sqrt{\frac{1}{2k-1}} g_{k+1}(f)(\mathcal{Y}), \quad (6.1)$$

$$H_k(f)(\mathcal{Y}) \leq \sqrt{\frac{1}{2k}} h_{k+1}(f)(\mathcal{Y}), \quad G_k(f)(\mathcal{Y}) \leq \sqrt{\frac{1}{2k}} g_{k+1}(f)(\mathcal{Y}), \quad (6.2)$$

and

$$g_{2k}(f)(\mathcal{Y}) \leq \sqrt{\frac{1}{2} \frac{(4k)!}{(2k)!}} h_k(f)(\mathcal{Y}). \quad (6.3)$$

Remark Recall that for a general symmetric semigroup with the contraction property, Stein showed in [20, p. 75] that

$$\sup_{t>0} |T_t f| \leq \sup_{t>0} \left| \frac{1}{t} \int_0^t T_s f \, ds \right| + \left(\int_0^{+\infty} \left| t \frac{\partial}{\partial t} T_t f \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \quad (6.4)$$

Proof of Proposition 7 We first verify that

$$\left| \frac{\partial^k}{\partial t^k} p_t^{(n, \alpha)}(\mathcal{Y}, \mathcal{Y}') \right| \lesssim y^{-\alpha} t^{-\frac{1}{2}} \left(1 + t^{-\frac{n-1}{2}-k} \right), \quad \forall t > 0, \quad k \in \mathbb{N}, \quad \mathcal{Y}, \mathcal{Y}' \in \mathbb{H}^n. \quad (6.5)$$

From Eqs. 5.1, 5.18 and 5.4, we get

$$\begin{aligned} \left| \frac{\partial^k}{\partial t^k} p_t^{(n,\alpha)}(\mathcal{Y}, \mathcal{Y}') \right| &\lesssim (yy')^{-\alpha/2} \left[1 + \frac{1}{t^k} + \left(\frac{r}{t} \right)^{2k} \right] \\ &\quad \times t^{-\frac{1}{2}} \frac{1+r}{t} \left(1 + \frac{1+r}{t} \right)^{\frac{n-3}{2}} \\ &\quad \times \exp \left(-(\rho(n) + \rho(n, \alpha))r - t \left(\frac{r}{2t} - \rho(n, \alpha) \right)^2 \right). \quad (6.6) \end{aligned}$$

Since $y/2y' \leq \cosh r$ and $\rho(n) + \rho(n, \alpha) \geq \alpha/2$, we have

$$(yy')^{-\alpha/2} \exp(-(\rho(n) + \rho(n, \alpha))r) \lesssim y^{-\alpha} (\cosh r)^{\alpha/2} \exp\left(-\frac{\alpha r}{2}\right) \lesssim y^{-\alpha}.$$

If $r/t \lesssim 1$, then Eq. 6.5 is immediate from this and Eq. 6.6. If r/t is large, we will get in Eq. 6.6 a factor $\exp(-cr^2/t)$ with some $c > 0$. It allows us to replace r in the polynomial factors in Eq. 6.6 by \sqrt{t} , and Eq. 6.5 follows again.

For $f \in L^1$ and $\mathcal{Y} \in \mathbb{H}^n$, the estimate (6.5) implies that

$$\lim_{t \rightarrow +\infty} \frac{\partial^k}{\partial t^k} e^{t\Delta_{\mathbb{H}(n,\alpha)}} f(\mathcal{Y}) = 0.$$

We claim that also

$$\lim_{t \rightarrow +\infty} \frac{\partial^k}{\partial t^k} e^{-t\sqrt{-\Delta_{\mathbb{H}(n,\alpha)}}} f(\mathcal{Y}) = 0.$$

Indeed, with $P_t^{(n,\alpha)}(\mathcal{Y}, \mathcal{Y}')$ denoting the Poisson kernel, the subordination formula (4.1) implies that for $t > 1$

$$\begin{aligned} \left| \frac{\partial^k}{\partial t^k} P_t^{(n,\alpha)}(\mathcal{Y}, \mathcal{Y}') \right| &\leq \int_0^{+\infty} \frac{1}{u} \left(\frac{1}{2\sqrt{u}} \right)^k \left| \phi^{(k)} \left(\frac{t}{2\sqrt{u}} \right) \right| p_u^{(n,\alpha)}(\mathcal{Y}, \mathcal{Y}') du \\ &\lesssim y^{-\alpha} \int_0^{+\infty} \frac{1}{u} \left(\frac{1}{2\sqrt{u}} \right)^k \left| \phi^{(k)} \left(\frac{t}{2\sqrt{u}} \right) \right| u^{-\frac{1}{2}} (1 + u^{-\frac{n-1}{2}}) du \\ &\lesssim y^{-\alpha} t^{-k-1}. \end{aligned}$$

Following the argument in [21, p. 86], one obtains immediately Eqs. 6.1 and 6.2. For example,

$$\frac{\partial^k}{\partial t^k} e^{t\Delta_{\mathbb{H}(n,\alpha)}} f(\mathcal{Y}) = - \int_t^{+\infty} \frac{\partial^{k+1}}{\partial s^{k+1}} e^{s\Delta_{\mathbb{H}(n,\alpha)}} f(\mathcal{Y}) s^{k+\frac{1}{2}} \frac{ds}{s^{k+\frac{1}{2}}},$$

and the Cauchy-Schwarz inequality leads to $H_k(f)(\mathcal{Y})^2 \leq \frac{1}{2k} h_{k+1}(f)(\mathcal{Y})^2$. For (6.1), one can also use Hardy's inequality, which actually gives a better constant, $1/k$ instead of $1/\sqrt{2k-1}$.

The inequality (6.3) is valid in a general situation. We prove it by adapting the method in [3, p. 52]. Indeed,

$$g_{2k}^2(f)(\mathcal{Y}) = \int_0^{+\infty} t^{4k-1} \left| (-\Delta)^k e^{-t\sqrt{-\Delta}} f(\mathcal{Y}) \right|^2 dt.$$

The subordination formula (2.3) implies

$$g_{2k}^2(f)(\mathcal{Y}) \leq \frac{1}{\pi} \int_0^{+\infty} t^{4k-1} \left(\int_0^{+\infty} \left| (-\Delta)^k e^{\frac{t^2}{4u}\Delta} f(\mathcal{Y}) \right| e^{-u} u^{-\frac{1}{2}} du \right)^2 dt.$$

We estimate the inner integral here by using the Cauchy-Schwarz inequality, getting

$$\begin{aligned} g_{2k}^2(f)(\mathcal{Y}) &\leq \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{4k-1} \left(\int_0^{+\infty} \left| (-\Delta)^k e^{\frac{t^2}{4u}} f(\mathcal{Y}) \right|^2 e^{-u} u^{-\frac{1}{2}} du \right) dt \\ &= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \left(\int_0^{+\infty} t^{4k-1} \left| (-\Delta)^k e^{\frac{t^2}{4u}} f(\mathcal{Y}) \right|^2 dt \right) e^{-u} u^{-\frac{1}{2}} du. \end{aligned}$$

The change of variable $\lambda = t^2/4u$ shows that the last inner integral equals $2^{-1}(4u)^{2k} h_k^2(f)(\mathcal{Y})$. As a result,

$$g_{2k}^2(f)(\mathcal{Y}) \leq \frac{1}{\sqrt{\pi}} 2^{-1} 4^{2k} \Gamma(2k+1/2) h_k^2(f)(\mathcal{Y}).$$

This easily implies Eq. 6.3. Proposition 7 is proved. \square

7 Weak Type (1, 1) Boundedness of h_1 , H_0 and g_k , $k \geq 1$

Since $\mathbb{H}^{(n,\alpha)}$ has the local doubling property as pointed out in Section 2, one can use the method of localization. Standard vector-valued singular integral operator theory then gives the weak type (1, 1) of the local part of h_1 .

Consider now the part at infinity of h_1 , given by

$$h_1^\infty(f)(\mathcal{Y}) = \left(\int_0^{+\infty} s \left| \int_{d(\mathcal{Y}, \mathcal{Y}') \geq 1} \frac{\partial}{\partial s} \left[(yy')^{-\frac{\alpha}{2}} K(n, \alpha; s, r) \right] f(\mathcal{Y}') d\mu_\alpha(\mathcal{Y}') \right|^2 ds \right)^{\frac{1}{2}}.$$

Using Minkowski's integral inequality, we get

$$h_1^\infty(f)(\mathcal{Y}) \leq \int_{r \geq 1} (yy')^{-\frac{\alpha}{2}} |f(\mathcal{Y}')| \left(\int_0^{+\infty} s \left| \frac{\partial}{\partial s} K(n, \alpha; s, r) \right|^2 ds \right)^{\frac{1}{2}} d\mu_\alpha(\mathcal{Y}').$$

According to Lemma 3, it suffices to show that

$$\left(\int_0^{+\infty} s \left| \frac{\partial}{\partial s} K(n, \alpha; s, r) \right|^2 ds \right)^{\frac{1}{2}} \lesssim e^{-\frac{n-1+|n-1-\alpha|}{2}r}, \quad r \geq 1.$$

Because of Eqs. 5.16 and 5.4, it remains to prove that for $r \geq 1$

$$\int_0^{+\infty} \left[\frac{r}{s} \left(1 + \frac{r}{s} \right)^{\frac{n-3}{2}} e^{-s(\frac{r}{2s} - \rho(n, \alpha))^2} \left(\left| \frac{r^2}{4s^2} - \rho(n, \alpha)^2 \right| + \frac{1}{s} \right) \right]^2 ds \lesssim 1. \quad (7.1)$$

By the change of variables $\lambda = \frac{r}{2s} - \rho(n, \alpha)$, the left-hand side here can be rewritten as

$$2r \int_{-\rho(n, \alpha)}^{+\infty} [1 + 2(\lambda + \rho(n, \alpha))]^{n-3} e^{-r \frac{\lambda^2}{\lambda + \rho(n, \alpha)}} \left[|\lambda| (\lambda + 2\rho(n, \alpha)) + 2 \frac{\lambda + \rho(n, \alpha)}{r} \right]^2 d\lambda.$$

We split this integral at the point $\lambda = \rho(n, \alpha)$, to get

$$2r \int_{-\rho(n, \alpha)}^{\rho(n, \alpha)} \lesssim r \int e^{-\frac{r}{2\rho(n, \alpha)} \lambda^2} \left(|\lambda| + \frac{1}{r} \right)^2 d\lambda \lesssim r r^{-3/2} + r \frac{1}{r^2} \frac{1}{\sqrt{r}} \lesssim 1$$

since $r \geq 1$, and

$$2r \int_{\rho(n,\alpha)}^{\infty} \lambda^{n-3} e^{-r\lambda/2} \lambda^4 d\lambda \lesssim r^{1-n-2} \lesssim 1.$$

We have verified (7.1) and thus the weak type $(1, 1)$ of h_1 .

The weak type $(1, 1)$ of H_0 follows from the above together with Eq. 6.4 and the Hopf-Dunford-Schwartz maximal ergodic theorem.

Finally we consider the weak type $(1, 1)$ of g_k , $k \geq 1$, proceeding as in the case of h_1 above. The local part causes no problems, and for the part at infinity, it is enough to verify the following estimate:

$$\left(\int_0^{+\infty} s^{2k-1} \left| \frac{\partial^k}{\partial s^k} P_s^{(n,\alpha)}(\mathcal{Y}, \mathcal{Y}') \right|^2 ds \right)^{\frac{1}{2}} \lesssim (yy')^{-\frac{\alpha}{2}} e^{-\frac{n-1+|n-1-\alpha|}{2}r} \quad \forall r \geq 1. \quad (7.2)$$

To this end, we use the subordination formula (4.1), the function ϕ from Eq. 4.2 and Minkowski's integral inequality, to write

$$\begin{aligned} & \left(\int_0^{+\infty} \left| s^k \frac{\partial^k}{\partial s^k} P_s^{(n,\alpha)}(\mathcal{Y}, \mathcal{Y}') \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\ & \leq (yy')^{-\frac{\alpha}{2}} \int_0^{+\infty} \left\{ \left(\int_0^{+\infty} \left| \left(\frac{s}{2\sqrt{u}} \right)^k \phi^{(k)} \left(\frac{s}{2\sqrt{u}} \right) \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\} \frac{1}{u} K(n, \alpha; u, r) du \\ & \lesssim (yy')^{-\frac{\alpha}{2}} \int_0^{+\infty} \frac{1}{u} K(n, \alpha; u, r) du. \end{aligned}$$

In view of Eq. 5.4, the last expression is majorized by

$$(yy')^{-\frac{\alpha}{2}} e^{-\frac{n-1+|n-1-\alpha|}{2}r} \int_0^{+\infty} u^{-\frac{3}{2}} \frac{r}{u} \left(1 + \frac{r}{u} \right)^{\frac{n-3}{2}} e^{-u(\frac{r}{2u} - \rho(n,\alpha))^2} du.$$

By the change of variable $u = rs$, we see that for $r \geq 1$ the last integral is no larger than

$$r^{-\frac{1}{2}} \int_0^{+\infty} s^{-\frac{3}{2}} \frac{1}{s} \left(1 + \frac{1}{s} \right)^{\frac{n-3}{2}} e^{-s(\frac{1}{2s} - \rho(n,\alpha))^2} ds \lesssim r^{-\frac{1}{2}}.$$

and Eq. 7.2 follows.

8 Weak Type $(1, 1)$ of H_1

For $f \in L^1$, we write, using Eq. 5.1,

$$\begin{aligned} t \frac{\partial}{\partial t} e^{t\Delta_{\mathbb{H}(n,\alpha)}} f(\mathcal{Y}) &= \int_{r \leq 1} (yy')^{-\frac{\alpha}{2}} t \frac{\partial}{\partial t} K(n, \alpha; t, r) f(\mathcal{Y}') d\mu_{\alpha}(\mathcal{Y}') \\ &\quad + \int_{r \geq 1} (yy')^{-\frac{\alpha}{2}} t \frac{\partial}{\partial t} K(n, \alpha; t, r) f(\mathcal{Y}') d\mu_{\alpha}(\mathcal{Y}'). \end{aligned} \quad (8.1)$$

For the first integral here, we claim that for any $t > 0$

$$\int_{r \leq 1} (yy')^{-\frac{\alpha}{2}} \left| t \frac{\partial}{\partial t} K(n, \alpha; t, r) f(\mathcal{Y}') \right| d\mu_{\alpha}(\mathcal{Y}') \lesssim M_{\text{loc}} f(\mathcal{Y}), \quad (8.2)$$

where the local maximal function is defined by

$$M_{\text{loc}} f(\mathcal{Y}) = \sup_{0 < r \leq 1} \frac{1}{\mu_{\alpha}(B(\mathcal{Y}, r))} \int_{B(\mathcal{Y}, r)} |f(\mathcal{Y}')| d\mu_{\alpha}(\mathcal{Y}').$$

Indeed, one can deduce from Eqs. 5.16 and 5.3 that for $r \leq 1$ and some $c > 0$

$$\left| t \frac{\partial}{\partial t} K(n, \alpha; t, r) \right| \lesssim \begin{cases} t^{-\frac{n}{2}} e^{-c \frac{r^2}{t}}, & \text{if } 0 < t \leq 1; \\ 1, & \text{if } t > 1. \end{cases}$$

Using for instance a local version of the argument in [8, (2.4) THE MAXIMAL THEOREM, pp. 63–64], one now obtains Eq. 8.2. The measure μ_{α} has the local doubling property, and therefore the operator M_{loc} is of weak type $(1, 1)$ with respect to μ_{α} .

It remains to deal with the second integral in Eq. 8.1, and we start with a lemma.

Lemma 8 For $r \geq 1$ and any $t > 0$,

$$\left| t \frac{\partial}{\partial t} K(n, \alpha; t, r) \right| \lesssim e^{-\frac{n-1+|n-1-\alpha|}{2} r}.$$

Proof From Eqs. 5.16 and 5.4, we see that it is enough to show that for such r, t

$$\left[1 + t \left| \frac{r^2}{4t^2} - \rho(n, \alpha)^2 \right| \right] t^{-\frac{1}{2}} \frac{r}{t} \left(1 + \frac{r}{t} \right)^{\frac{n-3}{2}} e^{-t(\frac{r}{2t} - \rho(n, \alpha))^2} \lesssim 1.$$

The left-hand side here can be rewritten as

$$\begin{aligned} & r^{-\frac{1}{2}} \left(\frac{r}{t+r} \right)^{\frac{3}{2}} \left(1 + \frac{r}{t} \right)^{\frac{n}{2}} e^{-t(\frac{r}{2t} - \rho(n, \alpha))^2} \\ & + \frac{r}{t+r} \left(1 + \frac{r}{t} \right)^{\frac{n-1}{2}} \left(\frac{r}{2t} + \rho(n, \alpha) \right) \sqrt{t} \left| \frac{r}{2t} - \rho(n, \alpha) \right| e^{-t(\frac{r}{2t} - \rho(n, \alpha))^2}. \end{aligned}$$

If $r \leq 4\rho(n, \alpha)t$, this quantity is immediately seen to be bounded. In the opposite case, one has $\frac{r}{2t} - \rho(n, \alpha) > \frac{r}{4t}$, and then the same quantity is no larger than constant times

$$r^{-\frac{1}{2}} \left(\frac{r}{t} \right)^{\frac{n}{2}} e^{-\frac{1}{16} \frac{r^2}{t}} + \left(\frac{r}{t} \right)^{\frac{n+1}{2}} \left(\frac{r}{\sqrt{t}} \right) e^{-\frac{1}{16} \frac{r^2}{t}}.$$

Since $r \geq 1$, this expression stays bounded, and the desired inequality follows also in this case. Lemma 8 is proved. \square

Because of this lemma, the absolute value of the second integral in Eq. 8.1 can be estimated by constant times

$$\int_{r \geq 1} (yy')^{-\frac{\alpha}{2}} e^{-\frac{n-1+|n-1-\alpha|}{2} r} |f(\mathcal{Y}')| d\mu_{\alpha}(\mathcal{Y}').$$

As a result,

$$H_1(f)(\mathcal{Y}) \lesssim M_{\text{loc}} f(\mathcal{Y}) + \int_{r \geq 1} (yy')^{-\frac{\alpha}{2}} e^{-\frac{n-1+|n-1-\alpha|}{2} r} |f(\mathcal{Y}')| d\mu_{\alpha}(\mathcal{Y}'), \quad \mathcal{Y} \in \mathbb{H}^n,$$

and Lemma 3 implies the weak type $(1, 1)$ of H_1 . \square

9 The Operators h_k and H_k ($k \geq 2$) are not Bounded from L^1 to $L^{1,\infty}$

Let us start by studying h_k . According to Eq. 6.1, it is enough to prove that h_2 is not of weak type $(1, 1)$. Let δ_e be a unit point mass at the origin $e = (1, 0)$. If the assertion does not hold, a standard limit argument would imply $h_2(\delta_e) \in L^{1,\infty}$.

Fix a large constant $C_0 > 0$, and consider a point $\mathcal{Y} \in \mathbb{H}^n$ with $r = d(\mathcal{Y}, e)$ large. Then

$$h_2(\delta_e)(\mathcal{Y}) \geq \left(\int_{|\frac{r}{2t} - \rho(n, \alpha)| \leq \frac{1}{C_0 \sqrt{r}}} t^3 \left| y^{-\frac{\alpha}{2}} \frac{\partial^2}{\partial t^2} K(n, \alpha; t, r) \right|^2 dt \right)^{\frac{1}{2}}.$$

Using Eqs. 5.17 and 5.4, we obtain

$$\begin{aligned} h_2(\delta_e)(\mathcal{Y}) &\gtrsim y^{-\frac{\alpha}{2}} \left(\int_{|\frac{r}{2t} - \rho(n, \alpha)| \leq \frac{1}{C_0 \sqrt{r}}} t^3 \left| \frac{1}{r} K(n, \alpha; t, r) \right|^2 dt \right)^{\frac{1}{2}} \\ &\gtrsim y^{-\frac{\alpha}{2}} e^{-\frac{n-1+|n-1-\alpha|}{2} r} \left(\int_{|\frac{r}{2t} - \rho(n, \alpha)| \leq \frac{1}{C_0 \sqrt{r}}} dt \right)^{\frac{1}{2}} \\ &\gtrsim r^{\frac{1}{4}} y^{-\frac{\alpha}{2}} e^{-\frac{n-1+|n-1-\alpha|}{2} r}. \end{aligned}$$

For $\alpha > n - 1$, we shall apply this inequality to points in the set

$$\Omega_\gamma = \{\mathcal{Y} = (y, x) : e^\gamma/2 < y < e^\gamma, |x| < e^\gamma\},$$

for large values of $\gamma > 0$. These points will satisfy $\cosh r \sim e^\gamma$ and $r \sim \gamma$, so that $h_2(\delta_e)(\mathcal{Y}) \gtrsim \gamma^{\frac{1}{4}} e^{-\alpha\gamma}$. Observe also that $\mu_\alpha(\Omega_\gamma) \sim e^{\alpha\gamma}$. When $\alpha < n - 1$, we consider instead the set

$$\omega_\gamma = \{\mathcal{Y} = (y, x) : e^{-\gamma} < y < 2e^{-\gamma}, |x| < 1\},$$

in which we get $h_2(\delta_e)(\mathcal{Y}) \gtrsim \gamma^{\frac{1}{4}} e^{(\alpha-n+1)\gamma}$. Further, $\mu_\alpha(\omega_\gamma) \sim e^{(n-1-\alpha)\gamma}$. In both cases, the weak type $(1, 1)$ inequality is violated if we let $\gamma \rightarrow +\infty$.

To deal with H_k , $k \geq 2$, we argue as above. Assuming that $\mathcal{Y} \in \mathbb{H}^n$ with $r = d(\mathcal{Y}, e)$ large, we shall show that

$$H_k(\delta_e)(\mathcal{Y}) \gtrsim r^{\frac{1}{4}} y^{-\frac{\alpha}{2}} e^{-\frac{n-1+|n-1-\alpha|}{2} r}.$$

We choose $C_0 > \max(4\rho(n, \alpha), \rho(n, \alpha)^{-1})$ and write

$$\begin{aligned} \left(y^{\frac{\alpha}{2}} H_k(\delta_e)(\mathcal{Y}) \right)^2 &\geq \sup_{\frac{r}{C_0} < t < C_0 r} \left| t^k \frac{\partial^k}{\partial t^k} K(n, \alpha; t, r) \right|^2 \\ &\gtrsim \int_{\frac{r}{C_0}}^{C_0 r} \left| t^k \frac{\partial^k}{\partial t^k} K(n, \alpha; t, r) \right|^2 \frac{dt}{t} \\ &= \left(y^{\frac{\alpha}{2}} h_k(\delta_e)(\mathcal{Y}) \right)^2 - \int_0^{\frac{r}{C_0}} \left| t^k \frac{\partial^k}{\partial t^k} K(n, \alpha; t, r) \right|^2 \frac{dt}{t} \\ &\quad - \int_{C_0 r}^{+\infty} \left| t^k \frac{\partial^k}{\partial t^k} K(n, \alpha; t, r) \right|^2 \frac{dt}{t} \\ &= \left(y^{\frac{\alpha}{2}} h_k(\delta_e)(\mathcal{Y}) \right)^2 - W_1 - W_2. \end{aligned}$$

Observe that Eq. 6.1 remains valid for $f = \delta_e$, so that

$$y^{\frac{\alpha}{2}} h_k(\delta_e)(\mathcal{Y}) \geq y^{\frac{\alpha}{2}} h_2(\delta_e)(\mathcal{Y}) \gtrsim r^{\frac{1}{4}} e^{-\frac{n-1+|n-1-\alpha|}{2}r}.$$

The result will follow if we show that

$$W_1 + W_2 \lesssim e^{-(n-1+|n-1-\alpha|)r}.$$

To estimate the integral W_2 , we apply Eq. 5.18 together with Eq. 5.4 and observe that here $|r/2t - \rho(n, \alpha)| > \rho(n, \alpha)/2$, by the choice of C_0 . The result is

$$W_2 \lesssim e^{-(n-1+|n-1-\alpha|)r} \int_{C_0 r}^{+\infty} t^{2k-2} e^{-t\rho(n, \alpha)^2/2} dt \lesssim e^{-(n-1+|n-1-\alpha|)r}.$$

Similarly, $|r/2t - \rho(n, \alpha)| > r/4t$ in W_1 , and

$$\begin{aligned} W_1 &\lesssim e^{-(n-1+|n-1-\alpha|)r} \int_0^{\frac{r}{C_0}} \left(t^k \left(\frac{r}{t} \right)^{2k} t^{-1} \left(\frac{r}{t} \right)^{1+(n-3)/2} \right)^2 e^{-r^2/8t} dt \\ &\lesssim e^{-(n-1+|n-1-\alpha|)r}. \end{aligned}$$

The argument for H_k , $k \geq 2$, is complete, and so is the proof of Theorem 1. \square

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